CERTAIN PROBLEMS OF TRANSIENT HEAT CONDUCTION IN WEDGES*

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Analyzed is the general methodology of solving problems in transient heat conduction in plane finite-length and infinitely long wedges under various boundary conditions with respect to heat transfer at the surfaces. Closed solutions are obtained for special cases.

Problems of transient heat conduction in an infinitely long wedge under constant boundary temperatures have been solved in [1-4]. With the aid of the Kantorovich–Lebedev integral transformation, the problem was analyzed in [6, 7] with boundary conditions of the first kind and the third kind with respect to heat transfer, but with the solutions limited in terms of the wedge angle and, besides, rather unwieldy for numerical computations. A similar problem was considered in [8], but there the author inaccurately formulated the boundary conditions of the third kind with respect to heat transfer at the wedge surfaces.

Very relevant, in the practical sense, is an analysis of transient temperature fields in a plane wedge when the temperatures at its boundaries are variable in time and in space. The stipulation of such boundary conditions makes it possible to analyze the thermal state of a wedge element under actual or very nearly actual conditions of heating and cooling. A stipulation of boundary conditions of the first kind with respect to heat transfer, i.e., a stipulation of boundary temperatures, makes it possible to correlate the calculated temperature field with thermocouple readings over the boundary surfaces. Thus, solutions obtained under boundary conditions with respect to heat transfer which vary in time and space can be applied directly to the analysis of the thermal state in a wedge specimen under actual heating and cooling conditions.

In this study the authors use the G. A. Grinberg method to arrive at a solution for a wedge with any angle and bounded by a circular arc of radius r = R, also for an infinitely long plane wedge where the boundary conditions of the third kind with respect to heat transfer vary with time and along the radius.

We consider a plane homogeneous wedge with an arbitrary angle α $(0 \le \varphi \le \alpha)$ and bounded by a circular arc of radius r = R $(0 \le r \le R)$ (Fig. 1). The temperatures at the wedge boundaries $\varphi = 0$ and $\varphi = \alpha$ are specified as known functions of the coordinate r and time τ, while the heat transfer at the boundary r = R to a medium at a zero reference temperature follows Newton's Law with $\varkappa$ denoting the relative coefficient of heat transfer. For simplicity, the initial temperature will be assumed constant and equal to t_0.

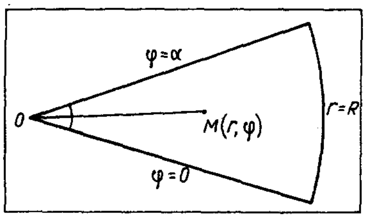

Fig. 1. Choice of coordinate system for a wedge bounded by a circular arc.

The problem of determining the temperature field of a bounded wedge reduces to integrating the differential equations of transient heat conduction [9]:

$$\frac{\partial t(r,\ \varphi,\ \tau)}{\partial \tau} = a^2\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 t}{\partial \varphi^2}\right) \tag{1}$$

with the initial conditions

$$t(r,\ \varphi,\ 0) = t_0 = \mathrm{const} \tag{2}$$

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and the boundary conditions

$$t(r,\ 0,\ \tau) = t_1(r,\ \tau),$$
$$t(r,\ \alpha,\ \tau) = t_2(r,\ \tau), \tag{3}$$

$$\frac{\partial t}{\partial r} + \varkappa t = 0 \quad \text{at} \quad r = R. \tag{4}$$

The condition of boundedness of the temperature $t(r,\ \varphi,\ \tau)$ at $r \to 0$ enters here naturally. The symbols in (1)-(4) are conventional [9].

We establish a system of eigenfunctions with respect to the angular coordinate φ, orthonormalized on the interval $[0,\ \alpha]$, which will represent the solution to the Sturm–Liouville homogeneous system in the form [5]

$$\Phi_m(\varphi) = \sqrt{\frac{2}{\alpha}} \sin \frac{(m+1)\pi}{\alpha} \varphi \qquad (m = 0,\ 1,\ 2,\ \ldots). \tag{5}$$

The solution to the original equation (1) will be expressed in terms of a series expansion with respect to eigenfunctions (5)

$$t(r,\ \varphi,\ \tau) = \sum_{m=0}^{\infty} t_m(r,\ \tau)\, \Phi_m(\varphi). \tag{6}$$

Then, for the coefficients in (6) we obtain the following differential equations

$$\frac{\partial t_m}{\partial \tau} = a^2 \left(\frac{\partial^2 t_m}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t_m}{\partial r} - \frac{\nu^2}{r^2} t_m \right) + F_m \tag{7}$$

with the initial condition

$$t_m(r,\ 0) = \sqrt{\frac{2}{\alpha}} \cdot \frac{t_0}{\nu} [1 + (-1)^m] \tag{8}$$

and the boundary condition on the circular arc $r = R$

$$\frac{\partial t_m}{\partial r} + \varkappa t_m = 0. \tag{9}$$

Here

$$\nu = \frac{(m+1)\pi}{\alpha},$$
$$F_m(r,\ \tau) = \sqrt{\frac{2}{\alpha}} \frac{a^2 \nu}{r^2} [t_1(r,\ \tau) + (-1)^m t_2(r,\ \tau)]. \tag{10}$$

For the boundary-value problem (7)-(9) we will use the finite Hankel integral transformation with respect to variable r, which is defined according to [10] as follows:

$$T_m(\lambda_k,\ \tau) = \int_0^R r t_m(r,\ \tau)\, J_\nu(\lambda_k r)\, dr, \tag{11}$$

with λ_k denoting the positive roots of the characteristic equation

$$\lambda J_\nu'(\lambda R) + \varkappa J_\nu(\lambda R) = 0. \tag{12}$$

The inverse transform of (11) is [10]

$$t_m(r,\ \tau) = \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R) + J_\nu'^2(\lambda_k R)} T_m(\lambda_k,\ \tau), \tag{13}$$

with summation over all roots of Eq. (12).

The use of the integral Eq. (11) in the problem (7)-(9) leads to a nonhomogeneous ordinary differential equation with respect to function $T_m(\lambda_k,\ \tau)$, which is solved by an inverse Hankel transformation according to (13) and yields the sought temperature field in a bounded wedge:

$$t(r,\ \varphi,\ \tau) = \sqrt{\frac{2}{\alpha}} \sum_{m=0}^{\infty} \sin \frac{(m+1)\pi}{\alpha} \varphi \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R) + J_\nu'^2(\lambda_k R)}$$
$$\times \left\{ \int_0^\tau \int_0^R \sqrt{\frac{2}{\alpha}} a^2 \nu \frac{1}{r} [t_1(r,\ u) + (-1)^m t_2(r,\ u)]\, J_\nu(\lambda_k r)\, dr \exp[-a^2 \lambda_k^2 (\tau\right.$$

$$-u)]\,du + \sqrt{\frac{2}{\alpha}}\cdot\frac{t_0}{\nu}[1+(-1)^m]\int_0^R rJ_\nu(\lambda_k r)\,dr\,\exp(-a^2\lambda_k^2\tau)\Bigg\}. \tag{14}$$

As an example, we will now consider the case of boundary temperatures varying linearly with time:

$$\begin{aligned} t(r,\ 0,\ \tau) &= t_1(r,\ \tau) = t_0 + v\tau,\\ t(r,\ \alpha,\ \tau) &= t_2(r,\ \tau) = t_0 + v\tau. \end{aligned} \tag{15}$$

For simplicity, we assume that heating occurs symmetrically with respect to the wedge angle bisector.

Performing the appropriate transformations by the described procedure, we obtain the temperature field of a bounded wedge under conditions (15):

$$t(r,\ \varphi,\ \tau) = \frac{2\sqrt{2}}{R^2\sqrt{\alpha}}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R)+J_\nu'^2(\lambda_k R)}\sin\frac{(m+1)\pi}{\alpha}\varphi$$

$$\times\left\{\frac{\nu Q_1(\lambda_k)}{\lambda_k^2}(t_0+v\tau) - \frac{v\nu Q_1(\lambda_k)}{a^2\lambda_k^4}[1-\exp(-a^2\lambda_k^2\tau] + \frac{\lambda_k^2 Q_2(\lambda_k)-\nu^2 Q_1(\lambda_k)}{\nu\lambda_k^2}t_0\exp(-a^2\lambda_k^2\tau)\right\}, \tag{16}$$

where

$$\begin{aligned} Q_1(\lambda_k) &= \sqrt{\frac{2}{\alpha}}[1+(-1)^m]\left[(\nu-2)\,\lambda_k R J_\nu(\lambda_k R)\,S_{-2,\nu-1}(\lambda_k R) - \lambda_k R J_{\nu-1}(\lambda_k R)\,S_{-1,\nu}(\lambda_k R) + \frac{1}{\nu}\right];\\ Q_2(\lambda_k) &= \sqrt{\frac{2}{\alpha}}[1+(-1)^m]\left[\frac{\nu R}{\lambda_k}J_\nu(\lambda_k R)\,S_{0,\nu-1}(\lambda_k R) - \frac{R}{\lambda_k}J_{\nu-1}(\lambda_k R)\,S_{1,\nu}(\lambda_k R) + \frac{\nu}{\lambda_k^2}\right]. \end{aligned} \tag{17}$$

Here $S_{x,y}(z)$ are Lommel functions [11, 12].

Of practical significance is the case where the boundary conditions at the wedge edges are given as

$$t_1(r,\ \tau) = t_2(r,\ \tau) = (t_0+v\tau)\left(1-\frac{cr^2}{R^2}\right), \tag{18}$$

i.e., when the boundary temperatures decrease parabolically (second-degree parabola with the parameter c) along the radius (with distance from the wedge point) and linearly with time. The temperature field in this case will be

$$t(r,\ \varphi,\ \tau) = \frac{2\sqrt{2}}{R^2\sqrt{\alpha}}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R)+J_\nu'^2(\lambda_k R)}\sin\nu\varphi$$

$$\times\left\{\frac{\nu[Q_1(\lambda_k)R^2-cQ_2(\lambda_k)]}{\lambda_k^2R^2}(t_0+v\tau) - \frac{v\nu[R^2Q_1(\lambda_k)-cQ_2(\lambda_k)]}{a^2R^2\lambda_k^4}\left(1-e^{-a^2\lambda_k^2\tau}\right)\right.$$

$$\left. + \frac{(\lambda_k^2R^2+\nu^2c)\,Q_2(\lambda_k)-\nu^2R^2Q_1(\lambda_k)}{\nu\lambda_k^2R^2}t_0e^{-a^2\lambda_k^2\tau}\right\}, \tag{19}$$

where, as before, $Q_1(\lambda_k)$ and $Q_2(\lambda_k)$ are determined from (17).

The described method of solving transient heat conduction problems in wedges can be effectively applied to the solution of analogous problems involving an infinitely long plane wedge with variable boundary conditions of the first kind with respect to heat transfer. For the case of an infinitely long wedge, $R\to\infty$, modifications of this method consist only in performing the Hankel integral transformation (11) with the limit on variable r moved to infinity [11]:

$$T_m(\lambda,\ \tau) = \int_0^\infty rt_m(r,\ \tau)\,J_\nu(\lambda r)\,dr \tag{20}$$

and the inverse transformation

$$t_m(r,\ \tau) = \int_0^\infty \lambda T_m(\lambda,\ \tau)\,J_\nu(\lambda r)\,d\lambda. \tag{21}$$

Let now the boundary temperatures of a wedge vary linearly with time and exponentially along the radius r, i.e.,

$$\begin{aligned} t(r,\ 0,\ \tau) &= t_1(r,\ \tau) = (t_0+v\tau)\,e^{-\gamma r^2},\\ t(r,\ \alpha,\ \tau) &= t_2(r,\ \tau) = (t_0+v\tau)\,e^{-\gamma r^2}. \end{aligned} \tag{22}$$

The initial temperature will be assumed to decrease exponentially along the radius r:

$$t(r,\ \varphi,\ 0) = t_0 e^{-\gamma r^2}.$$

Then the temperature field of an infinitely long plane wedge under the given conditions will be

$$t(r,\ \varphi,\ \tau) = \frac{2}{\alpha} \sum_{m=0,2,4\ldots} \left\{ \frac{(t_0 + v\tau) e^{-\gamma r^2} \Gamma\left(\frac{\nu}{2}\right)}{4\nu^2 \Gamma(\nu)} A(\nu) \right.$$

$$- \frac{v r^2 e^{-\gamma r^2} \Gamma\left(\frac{\nu}{2}\right)}{16\nu a^2 \Gamma(\nu)} \left[\frac{A(\nu-2)}{(\nu-1)(\nu-2)} + \frac{2A(\nu)}{\nu^2(\nu-1)} + \frac{A(\nu+2)}{(\nu+1)(\nu+2)} \right]$$

$$- \frac{r^\nu \sqrt{\gamma}}{2^\nu a^2 \nu \Gamma(\nu)} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{2}+k\right) \Gamma\left(\frac{\nu}{2}+k\right)(4\gamma)^{\nu-1}}{k!\,(\nu+k)(4a^2\gamma\tau+1)^{\nu+k}} \left[t_0 \Phi\left(\nu+k-\frac{1}{2},\ \nu+1;\ -\frac{\gamma r^2}{4a^2\gamma\tau+1}\right) \right.$$

$$\left. - \frac{v\Gamma(\nu+k-1)(4a^2\gamma\tau+1)}{16\gamma^2 \Gamma(\nu+k)} \Phi\left(\nu+k-\frac{3}{2},\ \nu+1;\ -\frac{\gamma r^2}{4a^2\gamma\tau+1}\right)\right]$$

$$+ \frac{r^{2\nu} t_0 \Gamma\left(\frac{\nu}{2}\right) e^{-\frac{r^2}{4a^2\tau}}}{2^{\nu+2}\Gamma(2\nu)(a\sqrt{\tau})^\nu \left(\sqrt{4a^2\gamma\tau+1}\right)^{\nu+2}} \Phi\left(\frac{\nu}{2},\ \nu+1;\right. \tag{23}$$

$$\left.\left. \frac{r^2}{4a^2\tau(4a^2\gamma\tau+1)}\right)\right\} \sin\nu\varphi.$$

Here

$$A(\nu) = \frac{\Gamma(\nu-1)}{(\nu-1)\Gamma\left(\frac{\nu}{2}-1\right)} + \frac{2\nu\Gamma(\nu+1)}{(\nu^2-1)\Gamma\left(\frac{\nu}{2}\right)} + \frac{\Gamma(\nu+3)}{(\nu+1)\Gamma\left(\frac{\nu}{2}+1\right)},$$

$\Gamma(z)$ is the gamma function, and $\Phi(x,\ y;\ w)$ is the degenerated hypergeometrical function in [12].

The resulting expressions (16) and (19) for the temperature field of a bounded plane wedge and (23) for an infinitely long plane wedge comprise series of known tabulated functions and can, therefore, be used for analyzing the thermal state of a wedge under given boundary conditions with respect to heat transfer.

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